TRINITY COLLEGE

Year 12 Mathematics Specialist 3/4
Test 3 2022

Section 1 Calculator Free

Vectors in Three Dimensions

STUDENT'S NAME Solutions [PRESSER]

DATE: Tuesday 17 May **TIME:** 25 minutes **MARKS:** 25

INSTRUCTIONS:

Standard Items: Pens, pencils, drawing templates, eraser

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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1. (4 marks)

Solve the following system of linear equations and state the geometric interpretation of the result.

$$x+2y-12z=-10$$
$$-x-3y-4z=5$$
$$4y-4z=-14$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -12 & -10 \\ -1 & -3 & -4 & 5 \\ 0 & 4 & -4 & -14 \end{array}\right]$$

$\therefore$ planes intersect at pt $(2, -3, \frac{1}{2})$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -12 & -10 \\ 0 & -1 & -16 & -5 \\ 0 & 2 & -2 & -7 \end{array}\right] \begin{array}{l} \\ R_1+R_2 \\ \frac{1}{2}R_3 \end{array}$$

✓ method / elim 1 variable

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -12 & -10 \\ 0 & -1 & -16 & -5 \\ 0 & 0 & -34 & -17 \end{array}\right] 2R_2+R_3$$

✓ first variable

✓ all variables

✓ reason/interp

$\Rightarrow \quad z = \frac{1}{2}$

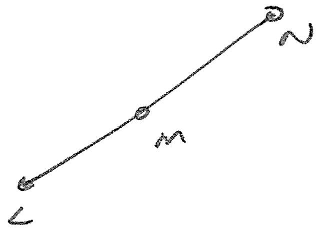
$\Rightarrow \quad -y - 8 = -5 \quad \Rightarrow y = -3$

$\Rightarrow \quad x - 6 - 6 = -10 \quad \Rightarrow \quad x = 2$

2. (8 marks)

Consider three points in space, $L\ (2,-1,4)$, $M\ (0,-17,10)$ and $N\ (5,23,-5)$.

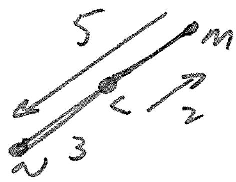
(a) Show that all three points are collinear. [3]



$$\overrightarrow{LM} = \begin{pmatrix} -2 \\ -16 \\ 6 \end{pmatrix} = -2\begin{pmatrix} 1 \\ 8 \\ -3 \end{pmatrix} \qquad \overrightarrow{MN} = \begin{pmatrix} 5 \\ 40 \\ -15 \end{pmatrix} = 5\begin{pmatrix} 1 \\ 8 \\ -3 \end{pmatrix}$$

So, $\overrightarrow{LM}$ and $\overrightarrow{MN}$ are parallel to $\begin{pmatrix} 1 \\ 8 \\ -3 \end{pmatrix}$ and they both pass through pt M.

$\therefore$ All three pts are collinear



(b) Complete the following statement by filling in the blanks. [2]

Point __L__ internally divides the line segment __$\overline{MN}$__ in the ratio __2__ : __3__.

(c) Determine the vector equation of the line $\overleftrightarrow{LM}$. [2]

$$\overrightarrow{LM} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 8 \\ -3 \end{pmatrix}$$

3. (4 marks)

A sphere has a centre $(0,4,0)$. Point $(4,16,3)$ lies on the sphere.

(a) Determine the vector equation of the sphere. [2]

$$\underset{\sim}{rad} = \begin{pmatrix} 4 \\ 16 \\ 3 \end{pmatrix} - \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 4 \\ 12 \\ 3 \end{pmatrix}$$

$$|\underset{\sim}{rad}| = \sqrt{16+144+9}$$

$$= \sqrt{169}$$

$$= 13$$

$$\therefore \quad \left| \underset{\sim}{r} - \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix} \right| = 13$$

(b) Determine the exact radius of the circle of the point of intersections between the sphere and the plane $y=14$. [2]

pts of intersection $\begin{pmatrix} x \\ 14 \\ z \end{pmatrix}$

Subbing this into the sphere

$$\Rightarrow \left| \begin{pmatrix} x \\ 14 \\ z \end{pmatrix} - \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix} \right| = 13$$

$$\Rightarrow \sqrt{x^2 + 100 + z^2} = 13$$

$$\Rightarrow x^2 + z^2 = 13^2 - 100$$

$$\Rightarrow x^2 + z^2 = 69$$

$\therefore$ radius of circle is $\sqrt{69}$

4. (9 marks)

An ant has a position defined by vector $\underset{\sim}{r}_a = \begin{pmatrix} 10 \\ 2 \\ n \end{pmatrix} + t_2 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. A beetle has a position defined by vector $\underset{\sim}{r}_b = \begin{pmatrix} 29 \\ 15 \\ 5 \end{pmatrix} + t \begin{pmatrix} -3 \\ -1 \\ 4 \end{pmatrix}$

(a) Determine the value of n which allows the paths of the ant and beetle to cross. [3]

Same position, different time

$\Rightarrow$ $29 - 3t = 10 + t_2$ — ①
$15 - t = 2 + 2t_2$ — ②
$5 + 4t = n + 3t_2$ — ③

$\Rightarrow$ $58 - 6t = 20 + 2t_2$ ①×2
$15 - t = 2 + 2t_2$

$\Rightarrow$ $43 - 5t = 18$
$\underline{\underline{t = 5}}$

So ② $\Rightarrow$ $10 = 2 + 2t_2$
$\underline{\underline{t_2 = 4}}$

So ③ $\Rightarrow$ $25 = n + 12$
$\therefore \underline{\underline{n = 13}}$

(b) From the initial position, how long should the ant wait before it starts moving to collide with the beetle? [1]

Time difference is $= t - t_2$
$= 5 - 4$
$= 1$ sec

$\therefore$ ant should wait 1 second.

If the ant and the beetle are walking in the same plane

(c) Determine the vector equation of the plane of the surface that the beetle and ant are walking on, in the form:

(i) $\underset{\sim}{r} = \underset{\sim}{a} + \lambda \underset{\sim}{b} + \mu \underset{\sim}{c}$ [1]

$$\underset{\sim}{r} = \begin{pmatrix} 29 \\ 15 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} -3 \\ -1 \\ 4 \end{pmatrix}$$

Use the pt you are given to minimise errors

(ii) $\underset{\sim}{r} \cdot \underset{\sim}{n} = k$ [3]

$$\underset{\sim}{n} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \times \begin{pmatrix} -3 \\ -1 \\ 4 \end{pmatrix}$$

$$\begin{array}{ccccc} i & j & k & i & j \\ 1 & 2 & 3 & 1 & 2 \\ -3 & -1 & 4 & -3 & -1 \end{array}$$

$$= \begin{pmatrix} 8+3 \\ -9-4 \\ -1+6 \end{pmatrix}$$

$$= \begin{pmatrix} 11 \\ -13 \\ 5 \end{pmatrix}$$

Easier pt

Now $\underset{\sim}{r} \cdot \begin{pmatrix} 11 \\ -13 \\ 5 \end{pmatrix} = \begin{pmatrix} 11 \\ -13 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 10 \\ 2 \\ 13 \end{pmatrix}$

$= 110 - 26 + 65$

$\therefore \underset{\sim}{r} \cdot \begin{pmatrix} 11 \\ -13 \\ 5 \end{pmatrix} = 149$

(d) Hence, state the cartesian equation of the plane. [1]

$\Rightarrow 11x - 13y + 5z = 149$

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Section 2 Calculator Assumed
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STUDENT'S NAME ______________________________

DATE: Tuesday 17 May **TIME:** 25 minutes **MARKS**: 25

INSTRUCTIONS:

Standard Items:	Pens, pencils, drawing templates, eraser
Special Items:	Three calculators, notes on one side of a single A4 page (these notes to be handed in with this assessment)

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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5. (2 marks)

Consider the following set of linear equations.

$$\begin{aligned} 2x+4y-z&=3 \quad \text{— (1)} \\ 3x+4y+z&=5 \quad \text{— (2)} \\ 4x+8y-2z&=5 \quad \text{— (3)} \end{aligned}$$

State and geometrically explain the solution to the system of linear equations.

No solution (calculator)

Planes (1) and (3) are parallel.

6. (10 marks)

Consider the vector functions of the displacement of two aircraft in meters, where t represents the number of seconds after the initial positional snapshot.
All answers should be rounded to two decimal places where appropriate.

Aircraft A $\underset{\sim}{r}_A = \begin{pmatrix} -3t+7 \\ 4t-1 \\ -0.1t^2+16 \end{pmatrix}$ and Aircraft B $\underset{\sim}{r}_B = \begin{pmatrix} 2t-14 \\ 3t+5 \\ -0.25t^2+2t+16 \end{pmatrix}$

(a) Determine the speed of Aircraft A three seconds after the positional snapshot. [3]

$$\underset{\sim}{v_a} = \begin{pmatrix} -3 \\ 4 \\ -0.2t \end{pmatrix}$$

$$\underset{\sim}{v_a}(3) = \begin{pmatrix} -3 \\ 4 \\ -0.6 \end{pmatrix}$$

$$\text{Speed} = |\underset{\sim}{v_a}| = 5.0359 \text{ m/s}$$
$$\approx 5.04 \text{ m/s}$$

(b) Determine the closest that these aircraft fly to each other, and state when this occurs. [3]

$$\text{let } \underset{\sim}{d} = \underset{\sim}{r_a} - \underset{\sim}{r_b}$$

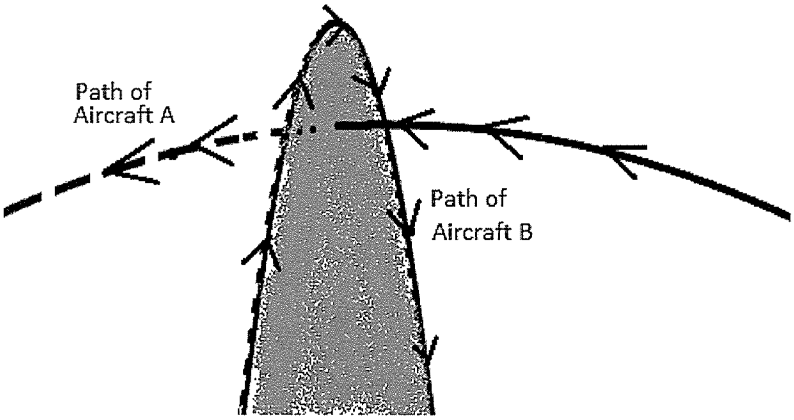
$$= \begin{pmatrix} -5t+21 \\ t-6 \\ 0.15t^2-2t \end{pmatrix}$$

using calculator $\text{fmin}(\text{norm}(\underset{\sim}{d}))$

$$\Rightarrow t = 4.10 \text{ sec}$$

$$\text{dist} = 6.01 \text{ m}$$

During the flight, Aircraft B dropped heavy smoke which quickly fell to the ground, creating a parabolic plane of smoke in the sky.



(c) considering aircraft B had a parabolic flight strictly along the plane $\underset{\sim}{r} \cdot \begin{pmatrix} -1.5 \\ 1 \\ 0 \end{pmatrix} = 26$.

(i) Determine the time when Aircraft A flew through the plane of smoke. [3]

$$A: \begin{pmatrix} -3t+7 \\ 4t-1 \\ -0.1t^2+16 \end{pmatrix} \cdot \begin{pmatrix} -1.5 \\ 1 \\ 0 \end{pmatrix} = 26$$

$$\Rightarrow \quad t = 4.41 \text{ sec}$$

(ii) State the coordinates where this occurred. [1]

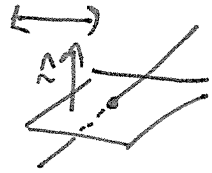
$$\underset{\sim}{r}_a(4.41) = \begin{pmatrix} -6.23 \\ 16.64 \\ 14.05 \end{pmatrix}$$

$\therefore$ pt $(-6.23, 16.64, 14.05)$

7. (6 marks)

Consider the plane with equation $\underset{\sim}{r} \cdot \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} = 3$, and a line with equation $\underset{\sim}{r} = \begin{pmatrix} 1 \\ 2 \\ n \end{pmatrix} + \lambda \begin{pmatrix} 8 \\ 1 \\ m \end{pmatrix}$

Determine values for n and m for there to be:



(a) a single point of intersection between the line and plane. [3]

Single point of intersection $\Rightarrow$ line not parallel to plane

$\Rightarrow$ line $\cdot \underset{\sim}{n} \neq 0$

$$\begin{pmatrix} 8 \\ 1 \\ m \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} \neq 0$$

$$m \neq 20$$

line can be positioned anywhere in space

$\Rightarrow$ $n \in \mathbb{R}$

(b) no points of intersection between the line and plane. [3]

No points $\Rightarrow$ line parallel

$\Rightarrow$ $m = 20$

But, line can't be in the plane

$$\Rightarrow \begin{pmatrix} 1+8\lambda \\ 2+\lambda \\ n+20\lambda \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} \neq 3$$

$\Rightarrow$ $n \neq 7$

8. (7 marks)

Consider a particle p with position vector $\underset{\sim}{r}_p(t) = \begin{pmatrix} a\cos(bt) \\ a\sin(bt) \end{pmatrix}$, $a > 0$.

(a) Prove that the particle moves under uniform circular motion [2]

$\underset{\sim}{v}_p(t) = \begin{pmatrix} -ab\sin(bt) \\ ab\cos(bt) \end{pmatrix}$

$\underset{\sim}{a}_p(t) = \begin{pmatrix} -ab^2\cos(bt) \\ -ab^2\sin(bt) \end{pmatrix}$

$= -b^2 \underset{\sim}{r}_p(t)$

Now $\underset{\sim}{v}_p(t) \cdot \underset{\sim}{a}_p(t)$

$= +a^2b^3\sin(bt)\cos(bt)$

$- a^2b^3\sin(bt)\cos(bt)$

$= 0$

$\therefore$ Circular motion

(b) State the period of motion [1]

Period $T = \frac{2\pi}{b}$

(c) Using part (b) and the fact that the distance travelled by a particle is given by the integral $\int_a^b |\underset{\sim}{v}(t)|\, dt$, Prove that the circumference of a circle can be found by the equation $C = 2\pi r$, where r is the radius of the circle. [4]

$C = \text{dist}$

$= \int_0^{2\pi/b} |\langle -ab\sin bt,\ ab\cos(bt) \rangle|\, dt$

$= \int_0^{2\pi/b} \sqrt{a^2b^2(\sin^2 bt + \cos^2 bt)}\, dt$

$= \int_0^{2\pi/b} ab\, dt$

$= [abt]_0^{2\pi/b}$

$= ab\frac{2\pi}{b}$

$= 2\pi a$

but $a = \text{radius}$

$\therefore C = 2\pi r$